

A NOTE ON RILEY POLYNOMIALS OF 2-BRIDGE KNOTS

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ABSTRACT. In this short note we show the existence of an epimorphism between groups of 2-bridge knots by means of an elementary argument using the Riley polynomial. As a corollary, we give a classification of 2-bridge knots by Riley polynomials.

1. INTRODUCTION

Let $K = S(\alpha, \beta)$ be a 2-bridge knot given by the Schubert normal form (see [1, Chapter 12]). Here $\alpha > 0$ and β are relatively prime odd integers satisfying $-\alpha < \beta < \alpha$. We denote the set of all such pairs (α, β) by \mathcal{S} . Two knots $S(\alpha, \beta)$ and $S(\alpha', \beta')$ are equivalent if and only if $\alpha = \alpha'$ and $\beta' \equiv \beta^{\pm 1} \pmod{\alpha}$. In this note, we will identify $S(\alpha, \beta)$ with its mirror image $S(\alpha, \beta)^* = S(\alpha, -\beta)$. Namely we will consider two 2-bridge knots $S(\alpha, \beta)$ and $S(\alpha', \beta')$ to be equivalent if and only if $\alpha = \alpha'$ and either $\beta' \equiv \beta^{\pm 1} \pmod{\alpha}$ or $\beta' \equiv -\beta^{\pm 1} \pmod{\alpha}$.

The knot group $G(K) = \pi_1(S^3 \setminus K)$ of $K = S(\alpha, \beta)$ has a presentation

$$G(K) = \langle x, y \mid wx = yw \rangle$$

where $w = x^{\epsilon_1} y^{\epsilon_2} \cdots x^{\epsilon_{\alpha-2}} y^{\epsilon_{\alpha-1}}$ and $\epsilon_i = (-1)^{\lfloor \frac{\beta}{\alpha} i \rfloor}$. Here we write $[r]$ for the greatest integer less than or equal to $r \in \mathbb{R}$. It is easily checked that $\epsilon_i = \epsilon_{\alpha-i}$ holds. The above presentation is not unique for a 2-bridge knot K itself, but the existence of at least one such presentation follows from Wirtinger's algorithm applied to the Schubert normal form of $S(\alpha, \beta)$. The generators x and y come from the two overpasses and present the meridian of $S(\alpha, \beta)$ up to conjugation.

A representation $\rho : G(K) \rightarrow SL(2; \mathbb{C})$ is called *parabolic* if $\text{tr} \rho(x) = \text{tr} \rho(y) = 2$ holds and ρ is nonabelian (i.e. $\text{Im } \rho$ is a nonabelian subgroup of $SL(2; \mathbb{C})$). We consider a map ρ from $\{x, y\}$ to $SL(2; \mathbb{C})$ given by

$$x \mapsto X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad y \mapsto Y = \begin{pmatrix} 1 & 0 \\ -u & 1 \end{pmatrix},$$

where $u \neq 0 \in \mathbb{C}$. Moreover we define a matrix $W = (w_{ij})$ by $W = \prod_{l=1}^{(\alpha-1)/2} X^{\epsilon_{2l-1}} Y^{\epsilon_{2l}}$. Riley proved in [6, Theorem 2] that the above map ρ

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gives a parabolic representation of $G(K)$ if and only if $w_{11} = 0$ holds. It is easy to see that any parabolic representation of $G(K)$ can be realized as the above form up to conjugation. We call this polynomial $w_{11} \in \mathbb{Z}[u]$ the *Riley polynomial* of $K = S(\alpha, \beta)$ and denote it by $\phi_{S(\alpha, \beta)}(u)$. Then we have a map $\mathcal{R} : \mathcal{S} \rightarrow \mathbb{Z}[u]$ defined by the correspondence $(\alpha, \beta) \mapsto \phi_{S(\alpha, \beta)}(u)$.

The purpose of the present note is to show the following result.

Theorem 1.1. *Let $K_1 = S(\alpha_1, \beta_1)$, $K_2 = S(\alpha_2, \beta_2)$ be 2-bridge knots. If $\phi_{S(\alpha_1, \beta_1)}(u) = \phi_{S(\alpha_2, \beta_2)}(u)$, then there exists an isomorphism $G(K_1) \rightarrow G(K_2)$ and thus K_1 and K_2 are equivalent.*

Remark 1.2. As we will see in the next section, the Riley polynomial is not determined by the knot group itself (see Example 2.1). Hence the converse statement of Theorem 1.1 does not hold. In other words, the map $\mathcal{R} : \mathcal{S} \rightarrow \mathbb{Z}[u]$ does not descend to the quotient \mathcal{S}/\sim , the set of equivalence classes of 2-bridge knots.

The above theorem directly follows from Theorem 3.1 which states the existence of an epimorphism between groups of 2-bridge knots. In the next section, we quickly review some properties of the Riley polynomial. We will prove Theorem 1.1 in Section 3. In the last section, we give a proof of a special case of Hartley-Murasugi's result to make this note as self-contained as possible.

2. PROPERTIES OF RILEY POLYNOMIALS

The definition of the Riley polynomial $\phi_{S(\alpha, \beta)}(u)$ depends on a choice of a presentation of the knot group $G(K)$. Namely $G(K)$ itself does not determine $\phi_{S(\alpha, \beta)}(u)$.

Example 2.1. Let us consider a pair of equivalent 2-bridge knots $K = S(7, 3)$ and $K' = S(7, 5)$. They have the isomorphic knot groups $G(K) \cong G(K')$ but different ϵ -sequences

$$(\epsilon_i) = (1, 1, -1, -1, 1, 1) \quad \text{and} \quad (\epsilon'_i) = (1, -1, 1, 1, -1, 1).$$

By a straightforward calculation, we have

$$\phi_{S(7, 3)}(u) = 1 - 2u + u^2 - u^3 \quad \text{and} \quad \phi_{S(7, 5)}(u) = 1 - 2u - 3u^2 - u^3.$$

Namely it shows that $\phi_{S(7, 3)}(u) \neq \phi_{S(7, 5)}(u)$ although $K = S(7, 3)$ and $K' = S(7, 5)$ are equivalent.

On the other hand, a 2-bridge knot $S(\alpha, \beta)$ and its mirror image $S(\alpha, \beta)^* = S(\alpha, -\beta)$ share the Riley polynomial.

Claim 2.2. *Let $K = S(\alpha, \beta)$ and $K^* = S(\alpha, -\beta)$ its mirror image. Then we have $\phi_{S(\alpha, -\beta)}(u) = \phi_{S(\alpha, \beta)}(u)$.*

Proof. Since $[r] + [-r] = -1$ holds for $r \in \mathbb{R} \setminus \mathbb{Z}$, ϵ -sequences for K and K^* satisfy $\epsilon_i^* = -\epsilon_i$ for $1 \leq i \leq \alpha - 1$. By the inductive argument on the length of ϵ -sequence, we can show that

$$w_{11}^* = w_{11}, w_{12}^* = -w_{12}, w_{21}^* = -w_{21}, w_{22}^* = w_{22}$$

for $W = (w_{ij}) = \prod_{l=1}^{(\alpha-1)/2} X^{\epsilon_{2l-1}} Y^{\epsilon_{2l}}$ and $W^* = (w_{ij}^*) = \prod_{l=1}^{(\alpha-1)/2} X^{\epsilon_{2l-1}^*} Y^{\epsilon_{2l}^*}$. Therefore we have $\phi_{S(\alpha, -\beta)}(u) = \phi_{S(\alpha, \beta)}(u)$. \square

Let \mathcal{S}_+ be the subset of \mathcal{S} satisfying $\beta > 0$ and further

$$\mathcal{S}_+^* = \{(\alpha, \beta) \in \mathcal{S}_+ \mid \exists (\alpha, \beta') \in \mathcal{S}_+ \text{ s.t. } \beta\beta' \equiv 1 \pmod{\alpha} \text{ and } \beta' < \beta\}.$$

We define $\overline{\mathcal{S}}$ to be $\mathcal{S}_+ \setminus \mathcal{S}_+^*$. As an immediate corollary of Theorem 1.1 and Claim 2.2, we have the following.

Corollary 2.3. *The restriction of \mathcal{R} , namely $\mathcal{R}|_{\overline{\mathcal{S}}} : \overline{\mathcal{S}} \rightarrow \mathbb{Z}[u]$ is injective.*

3. PROOF OF THEOREM 1.1

Let $K_i = S(\alpha_i, \beta_i)$ ($i = 1, 2$) be 2-bridge knots and $\phi_{S(\alpha_i, \beta_i)}(u)$ ($i = 1, 2$) their Riley polynomials. To prove Theorem 1.1, we first show the following.

Theorem 3.1. *If $\phi_{S(\alpha_2, \beta_2)}(u)$ is a factor of $\phi_{S(\alpha_1, \beta_1)}(u)$, then there exists an epimorphism from $G(K_1)$ to $G(K_2)$.*

Proof. Now we fix the presentations of $G(K_1)$ and $G(K_2)$ as in Section 1:

$$G(K_1) = \langle x_1, y_1 \mid w_1 x_1 = y_1 w_1 \rangle, \quad G(K_2) = \langle x_2, y_2 \mid w_2 x_2 = y_2 w_2 \rangle.$$

Assume $\phi_{S(\alpha_1, \beta_1)}(u) = \phi_{S(\alpha_2, \beta_2)}(u) \psi(u)$ for some $\psi(u) \in \mathbb{Z}[u]$, where the degree of $\phi_{S(\alpha_1, \beta_1)}(u)$ is $n + k$ and $\deg \phi_{S(\alpha_2, \beta_2)}(u) = n$. Let u_1, u_2, \dots, u_n be the zeros of $\phi_{S(\alpha_2, \beta_2)}(u)$ counting multiplicity. They give also zeros of $\phi_{S(\alpha_1, \beta_1)}(u)$ by the assumption.

Since each zero of the Riley polynomial corresponds to a parabolic representation of the knot group, we can consider all parabolic representations $\rho_1^1, \rho_2^1, \dots, \rho_{n+k}^1$ for $G(K_1)$ and $\rho_1^2, \rho_2^2, \dots, \rho_n^2$ for $G(K_2)$, where

$$\rho_j^1(x_1) = \rho_j^2(x_2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho_j^1(y_1) = \rho_j^2(y_2) = \begin{pmatrix} 1 & 0 \\ -u_j & 1 \end{pmatrix}$$

hold for $1 \leq j \leq n$. We take the direct products of these representations as

$$\begin{aligned} \Phi_1 : G(K_1) \ni g &\mapsto (\rho_j^1(g))_{j=1, \dots, n+k} \in SL(2; \mathbb{C})^{n+k}, \\ \Phi_2 : G(K_2) \ni g &\mapsto (\rho_j^2(g))_{j=1, \dots, n} \in SL(2; \mathbb{C})^n \end{aligned}$$

and put $\Gamma_i = \text{Im } \Phi_i$ which is a subgroup of $SL(2; \mathbb{C})^{n+k}$ or $SL(2; \mathbb{C})^n$ generated by $\Phi_i(x_i)$ and $\Phi_i(y_i)$. The natural projection to the first n factors $p : SL(2; \mathbb{C})^{n+k} \rightarrow SL(2; \mathbb{C})^n$ induces a homomorphism $\overline{p} : \Gamma_1 \rightarrow SL(2; \mathbb{C})^n$,

and clearly $\bar{p}(\Phi_1(x_1)) = \Phi_2(x_2)$ and $\bar{p}(\Phi_1(y_1)) = \Phi_2(y_2)$ hold. Hence we obtain an epimorphism $\bar{p} : \Gamma_1 \rightarrow \Gamma_2$.

By Thurston's hyperbolization theorem for Haken 3-manifolds, any knot in the 3-sphere S^3 is either a torus knot $T(p, q)$, or a satellite knot, or a hyperbolic knot, i.e. its complement admits a complete hyperbolic metric with finite volume (see [10]). By [8] a knot with bridge number 2 cannot be a satellite, hence a 2-bridge knot is a torus knot $T(2, q)$ where q is odd (because the bridge number of $T(p, q)$ is $\min\{|p|, |q|\}$ by [8]), or a hyperbolic knot. So let us consider the following two cases.

(1) If K_2 is hyperbolic, there exists a parabolic and faithful representation of $G(K_2)$ into $SL(2; \mathbb{C})$ (see [10]). Hence Φ_2 is injective. Applying the inverse of Φ_2 , we obtain an epimorphism

$$\Phi_2^{-1}|_{\Gamma_2} \circ \bar{p} \circ \Phi_1 : G(K_1) \rightarrow G(K_2).$$

This completes the proof in this case.

(2) If $K_2 = T(2, q)$, a torus knot with an odd integer q , there is a parabolic faithful representation $G(K_2)/Z(G(K_2)) \rightarrow PSL(2; \mathbb{R}) \subset PSL(2; \mathbb{C})$, where $Z(G(K_2)) \cong \mathbb{Z}$ is the center of $G(K_2)$ (see [9]). In fact, the image of $G(K_2)/Z(G(K_2))$ is isomorphic to the $(2, q, \infty)$ -triangle Fuchsian group in $PSL(2; \mathbb{R})$. This homomorphism lifts to $SL(2; \mathbb{C})$ and the lift is also faithful, so we can apply the same argument as in (1). Namely we have an epimorphism $G(K_1) \rightarrow G(K_2)/Z(G(K_2))$. Finally by a result of Hartley-Murasugi [3] (see also Section 4), this epimorphism lifts to $G(K_2)$. \square

Proof of Theorem 1.1. If $\phi_{S(\alpha_1, \beta_1)}(u) = \phi_{S(\alpha_2, \beta_2)}(u)$, we obtain two epimorphisms $\varphi_{12} : G(K_1) \rightarrow G(K_2)$ and $\varphi_{21} : G(K_2) \rightarrow G(K_1)$ by Theorem 3.1. Hence we have the epimorphism $\varphi_{21} \circ \varphi_{12} : G(K_1) \rightarrow G(K_1)$. Since any knot group $G(K)$ is known to be *Hopfian* (this follows from the facts that any knot group is residually finite [4] and a finitely generated, residually finite group is Hopfian [5]), namely, every epimorphism of $G(K)$ onto itself is an isomorphism, we see that φ_{12} and φ_{21} are injective. Therefore we can conclude $G(K_1) \cong G(K_2)$. Every 2-bridge knot is prime, so K_1 is equivalent to K_2 (see [2, Corollary 2.1]). This completes the proof.

4. LIFTING PROBLEMS

In this section, we prove that any epimorphism from any knot group onto the quotient of a torus knot group by its center can lift to the torus knot group. This is a special case of the result of Hartley and Murasugi (see [3]), but here we give a proof to make this note self-contained.

Let K be a knot and $G(K)$ its knot group with the abelianization $\alpha : G(K) \rightarrow \mathbb{Z} \cong \langle m \rangle$. We simply write $G(p, q)$ to $G(T(p, q))$ for the (p, q) -torus knot $T(p, q)$. It is known that $G(p, q)$ has the following presentation

$$G(p, q) = \langle x, y \mid x^p = y^q \rangle$$

and its center is the infinite cyclic group generated by $z = x^p = y^q$. We write

$$\pi : G(p, q) \rightarrow \overline{G}(p, q) = G(p, q) / \langle z \rangle = \langle x, y \mid x^p = y^q = 1 \rangle$$

and $\gamma : \mathbb{Z} = \langle m \rangle \ni m \mapsto \overline{m} \in \mathbb{Z}/pq = \langle \overline{m} \mid \overline{m}^{pq} = 1 \rangle$. Further we take the abelianization $\beta : \overline{G}(p, q) \ni x \mapsto \overline{m}^q, y \mapsto \overline{m}^p \in \mathbb{Z}/pq$.

Proposition 4.1 (Hartley-Murasugi [3]). *If $\overline{\varphi} : G(K) \rightarrow \overline{G}(p, q)$ is an epimorphism such that $\beta \circ \overline{\varphi} = \gamma \circ \alpha$, then there exists a lift $\varphi : G(K) \rightarrow G(p, q)$ of $\overline{\varphi} : G(K) \rightarrow \overline{G}(p, q)$ such that $\pi \circ \varphi = \overline{\varphi}$.*

Proof. The torus knot group $G(p, q) = \gamma^*(\overline{G}(p, q))$ can be described as the fiber product

$$\begin{array}{ccc} \gamma^*(\overline{G}(p, q)) & \xrightarrow{\tilde{\gamma} = \pi} & \overline{G}(p, q) \\ \tilde{\beta} \downarrow & & \downarrow \beta \\ \mathbb{Z} & \xrightarrow{\gamma} & \mathbb{Z}/pq. \end{array}$$

More precisely, the fiber product $\gamma^*(\overline{G}(p, q))$ is a subgroup of $\overline{G}(p, q) \times \mathbb{Z}$ as

$$\gamma^*(\overline{G}(p, q)) = \{(\overline{g}, m^s) \in \overline{G}(p, q) \times \mathbb{Z} \mid \beta(\overline{g}) = \gamma(m^s)\}.$$

Since there exists an exact sequence

$$1 \rightarrow \text{Ker}(\pi) = \langle (1, m^{pq}) \rangle \rightarrow \gamma^*(\overline{G}(p, q)) \rightarrow \overline{G}(p, q) \rightarrow 1,$$

we can see $G(p, q) \cong \gamma^*(\overline{G}(p, q))$.

By the assumption, we have an epimorphism $\overline{\varphi} : G(K) \rightarrow \overline{G}(p, q)$ such that $\beta \circ \overline{\varphi} = \gamma \circ \alpha$. For any $g \in G(K)$ we then define

$$\varphi(g) = (\overline{\varphi}(g), \alpha(g)) \in \overline{G}(p, q) \times \mathbb{Z},$$

and it gives an epimorphism $\varphi : G(K) \rightarrow G(p, q)$ satisfying $\pi \circ \varphi = \overline{\varphi}$. \square

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